

HYPERBOLIC 3-MANIFOLDS WITH k -FREE FUNDAMENTAL GROUP

ROSEMARY K. GUZMAN

ABSTRACT. The results of Culler and Shalen for 2, 3 or 4-free hyperbolic 3-manifolds are contingent on properties specific to and special about rank two subgroups of a free group. Here we determine what construction and algebraic information is required in order to make a geometric statement about M , a closed, orientable hyperbolic 3-manifold with k -free fundamental group, for any value of k greater than four. Main results are both to show what the formulation of the general statement should be, for which Culler and Shalen's result is a special case, and that it is true modulo a group-theoretic conjecture. A major result is in the $k = 5$ case of the geometric statement. Specifically, we show that the required group-theoretic conjecture is in fact true in this case, and so the proposed geometric statement when M is 5-free is indeed a theorem. One can then use the existence of a point P and knowledge about $\pi_1(M, P)$ resulting from this theorem to attempt to improve the known lower bound on the volume of M , which is currently 3.44 [8, Theorem 1.5].

1. INTRODUCTION

The goal of this paper is to explore how the geometry of a closed, orientable hyperbolic 3-manifold and its topological properties, especially its fundamental group, interact to provide new information about the manifold.

A hyperbolic n -manifold is a complete metric space that is locally isometric to the classical non-euclidean space \mathbf{H}^n in which the sum of the angles of a triangle is less than π , or, equivalently, a complete Riemannian manifold of constant sectional curvature -1 . Furthermore, one can express a hyperbolic n -manifold as the quotient of hyperbolic n -space modulo a discrete torsion-free group Γ of orientation-preserving isometries, in turn Γ is isomorphic to $\pi_1(M)$; it is this vantage point that we take in this paper.

We will say a group Γ is k -free, where k is a given positive integer, if every finitely generated subgroup of Γ of rank less than or equal to k is free. (Recall that the rank of a finitely generated group G is the minimal cardinality of a generating set for G .)

A recurring theme here is the interplay between classical topological properties of a hyperbolic 3-manifold and its geometric invariants, such as volume, and may even be regarded as a program for making the notion of Mostow Rigidity for hyperbolic 3-manifolds explicit. The property of having k -free fundamental group bridges these ideas via the $\log(2k - 1)$ -Theorem ([3, Main Theorem] combined with the Tameness Theorem of [1] and [5]), which uses geometric data about the manifold in regards to displacements of points under elements of $\pi_1(M)$ in \mathbf{H}^3 and forms the basis for the ideas of Section 2 of this paper.

One connection with topology is given by the first homology groups of M with coefficients in \mathbf{Z}_p : Given an integer $k \geq 3$ and M a closed, orientable, simple 3-manifold with the property that $\dim H_1(M; \mathbf{Z}_2) \geq \max(3k - 4, 6)$, then either $\pi_1(M)$ is k -free or M contains a closed, incompressible surface of genus at most $k - 1$ which is not a fibroid [6, Proposition 8.1].

Also, by a result of Jaco and Shalen in [9], any closed, orientable, hyperbolic 3-manifold M either satisfies the property that $\pi_1(M)$ is 2-free or has a finite cover, \widetilde{M} , with the rank of $\pi_1(\widetilde{M})$ equal to 2. In this paper we are concerned with the following geometric statement:

Geometric Conjecture 1.1. *If M is a closed, orientable, hyperbolic 3-manifold such that $\pi_1(M)$ is k -free with $k \geq 5$, then when $\lambda = \log(2k - 1)$, there exists a point P in M such that the set of all elements of $\pi_1(M, P)$ that are represented by loops of length less than λ is contained in a subgroup of $\pi_1(M)$ of rank $\leq k - 3$.*

In retrospect, results of Culler, Shalen, and Agol can be interpreted as special cases of this conjecture for the values $k = 3$ and $k = 4$; their work establishes those special cases of 1.1 in [2, Corollary 9.3] and [8, Theorem 1.4]. The present paper proves Conjecture 1.1 for the value $k = 5$, and also provides a method for showing what is required in general for the conjecture to hold for values of k greater than five. Cases $k \leq 4$ have further geometric consequences than the aforementioned connections suggest at first glance — for example, volume estimates for M to be mentioned below.

Our main result will relate the Geometric Conjecture of 1.1 to the following group-theoretic statement:

Group-Theoretic Conjecture 1.2. *Given two rank m subgroups of a free group whose intersection has rank greater than or equal to m , their join must have rank less than or equal to m ($m \geq 2$).*

This statement is the subject of Section 4 and was motivated by combining known results in the area as proved by Kent [10], Louder, and McReynolds [11]. In the $k = 4$ case of Conjecture 1.1, Culler and Shalen used Kent's result that if two rank-2 subgroups of a free group have rank-2 intersection, then they have a rank-2 join [10], but there were many details required to extend it to larger values of k .

The main result of this paper is the following theorem:

Implication Theorem 1.3. *Group-Theoretic Conjecture 1.2 with $m = k - 2$ implies Geometric Conjecture 1.1.*

After an introduction to some terminology in Section 2, Theorem 1.3 will be reformulated and proved as Theorem 5.5. In the proof, we consider the action of Γ on the sets of components of two disjoint subsets X_i, X_j of a simplicial complex K , and using [8, Lemma 5.12] and [8, Lemma 5.13], assuming the conclusion of the Theorem is false, we show that $\Gamma \leq \text{Isom}_+ \mathbf{H}^3$ admits a simplicial action without inversions on a tree $T = \mathcal{G}(X_i, X_j)$ with the property that the stabilizer in $\pi_1(M)$ of every vertex of T is a locally free subgroup of $\pi_1(M)$, which is a topological impossibility.

Following a suggestion of Marc Culler and using an argument in Kent's paper [10], we shall establish the validity of the Group-Theoretic Conjecture 1.2 for $m = 3$; this is the topic of Section 6. Hence, by Theorem 1.3, Geometric Conjecture 1.1 is established for the value of $k = 5$, and we have the following theorem:

Theorem 1.4. *Suppose M is a closed, orientable, hyperbolic 3-manifold such that $\pi_1(M)$ is k -free with $k = 5$. Then when $\lambda = \log 9$, there exists a point P in M such that the set of all elements of $\pi_1(M, P)$ that are represented by loops of length less than $\log 9$ is contained in a subgroup of $\pi_1(M)$ of rank ≤ 2 .*

As a corollary, we state some geometric properties for particular values of r_M in Section 6.

As mentioned above, there has been much work done to motivate the general statement of Theorem 1.3, and in those cases the geometric information was used to deduce lower bounds on the volume of M . Specifically, as a special case of Theorem 1.3, Culler and Shalen expressed the point P of the conclusion as a $\log 7$ -semithick point ([8, Theorem 1.4]), and using the existence of this point along with other consequences of 4-freeness, they were able to show that $\text{vol } M \geq 3.44$. In the 3-free case, Anderson, Canary, Culler and Shalen, along with Agol, showed the existence of a point P of M of injectivity radius $(\log 5)/2$ — this is exactly the point P described in Theorem 1.3 — and used its existence to establish that $\text{vol } M \geq 3.08$ in this case ([2, Corollary 9.3] and its predecessor [3, Theorem 9.1]).

Note that a closed hyperbolic manifold with k -free fundamental group, for $k \geq 2$, is in fact $k-1, k-2, k-3, \dots, 2$ -free. So, in particular, the results of Culler and Shalen [8] show that for a closed, orientable, hyperbolic 3-manifold M with 5-free fundamental group, we have $\text{vol } M \geq 3.44$. The method of obtaining this bound is by finding lower bounds for both the nearby volume of the $\log 7$ -semithick point P , i.e. the volume of the $(\log 7)/2$ neighborhood of P , and for the distant volume, i.e. the volume of the complement of this neighborhood. Therefore, a long range goal of the present work is to improve this bound with the added topological and geometric information that is gotten by virtue of the 5-free assumption and the rank ≤ 2 subgroup described in Theorem 1.4, with hopes that estimating the nearby and distant volumes of the given point P , under certain conditions, will lead to a refined lower bound on the global volume of M .

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2. LEMMA AND PRELIMINARIES

Definitions 2.1. Suppose we are given a positive real number $\lambda > 0$ and that the subgroup $\Gamma \leq \text{Isom}_+(\mathbf{H}^3)$ is discrete and cocompact (and so purely loxodromic). For $\gamma \in \Gamma$ we define the hyperbolic cylinder $Z_\lambda(\gamma)$ to be the set of points $P \in \mathbf{H}^3$ such that $d(P, \gamma \cdot P) < \lambda$.

Recall that since γ is loxodromic, there is a γ -invariant line, $A(\gamma) \subset \mathbf{H}^3$, called the *axis* of γ , such that γ acts on the points of $A(\gamma)$ as a translation by a distance $l > 0$, called the *translation length* of γ . For any point $P \in \mathbf{H}^3$, we have $d(P, \gamma \cdot P) \geq l$ with equality only when $P \in A(\gamma)$. Then as long as $l < \lambda$, the cylinder $Z_\lambda(\gamma)$ is non-empty (the radius of this cylinder is computed by a simple application of the hyperbolic law of cosines and is a monotonically increasing function for λ in the interval (l, ∞) ; see, for example, [7] for further details).

Remark 2.2. Given M a closed, orientable, hyperbolic 3-manifold, we may write M as the quotient \mathbf{H}^3/Γ , where Γ is a discrete group of orientation-preserving isometries of \mathbf{H}^3 that is torsion-free. Every isometry of Γ is loxodromic since M is closed (and so cannot be parabolic or elliptic). Every non-trivial element γ of Γ is contained in a unique maximal cyclic subgroup $C(\gamma)$ of Γ which is the centralizer of γ in Γ , which means that non-trivial elements of distinct maximal cyclic subgroups do not commute.

Definition 2.3. Supposing $M = \mathbf{H}^3/\Gamma$ is given as above, let $C(\Gamma)$ be the set of maximal cyclic subgroups of Γ . After fixing a positive real number λ , let $C_\lambda(\Gamma)$ denote the set of maximal cyclic subgroups $C = C(\gamma)$ of Γ having at least one (loxodromic) generator γ_0 of C with translation length less than λ .

Given a cyclic subgroup C of Γ , we define the set $Z_\lambda(C) = \bigcup_{1 \neq \gamma \in C} Z_\lambda(\gamma)$. Then if $C \in C_\lambda(\Gamma)$, the set $Z_\lambda(C)$ is in fact a cylinder of points in \mathbf{H}^3 that are displaced by a distance less than λ by some non-trivial element of C : specifically, there is a loxodromic element $\gamma \in C - \{1\}$ such that $Z_\lambda(C) = Z_\lambda(\gamma)$ (γ need not necessarily be γ_0 , the generator of C). Observe that if $C \in C(\Gamma) - C_\lambda(\Gamma)$, we have $Z_\lambda(C) = \emptyset$.

Note that the family of cylinders $(Z_\lambda(\gamma))_{1 \neq \gamma \in \Gamma}$ is *locally finite* as Γ is discrete; i.e. for every point P in \mathbf{H}^3 , there is a neighborhood of P which has non-empty intersection with only finitely many of the subsets $Z_\lambda(\gamma)$. Further, because the family $(Z_\lambda(\gamma))_{1 \neq \gamma \in \Gamma}$ is locally finite, so then is the family $(Z_\lambda(C))_{C \in C_\lambda(\Gamma)}$.

Remark 2.4. A locally finite family $\mathcal{Z} = (Z_\lambda(C))_{C \in C_\lambda(\Gamma)}$ of cylinders has a natural association to the set of maximal cyclic subgroups of Γ , and if this family of cylinders covers \mathbf{H}^3 , how it does so will be of particular importance, as we will later encode this information in the nerve (see Definition 3.1) of the cover \mathcal{Z} . Determining a “connectedness” argument for certain skeleta of the nerve in order to show homotopy-equivalence to \mathbf{H}^3 (and therefore contractibility), exhibited new challenges and many refinements in extending the 4-free arguments to the k -free arguments and are detailed in Section 3.

The following lemma is an application of the $\log(2k-1)$ Theorem ([3, Main Theorem] with [1] and [5]).

Lemma 2.5. *Suppose $\Gamma \leq \text{Isom}_+(\mathbf{H}^3)$ is discrete, loxodromic, k -free ($k \geq 2$) and torsion-free. If there exists a point $P \in Z_{\log(2k-1)}(C_1) \cap \cdots \cap Z_{\log(2k-1)}(C_n)$, then the rank of $\langle C_1, \dots, C_n \rangle$ is $\leq k-1$.*

Proof. (by induction on n)

Base case: If $n = 1$, then $P \in Z_{\log(2k-1)}(C)$. Because $\text{rk } C = 1$ and $k \geq 2$, $\text{rk } C \leq k - 1$ is satisfied.

Induction assumption: If $n = q$ then $X_q = \langle C_1, \dots, C_q \rangle$, and so we assume that $\text{rk } X_q \leq k - 1$.

Induction step: Notice that $X_{q+1} = \langle X_q, C_{q+1} \rangle = \langle C_1, \dots, C_q, C_{q+1} \rangle$. We must show that $\text{rk } X_{q+1} \leq k - 1$. To simplify notation, let $r = \text{rk } X_q$. First, consider when $\text{rk } \langle X_q, C_{q+1} \rangle = r$. Since $r \leq k - 1$ by our induction assumption, we are done.

Next, consider the case when $\text{rk } \langle X_q, C_{q+1} \rangle > \text{rk } X_q = r$.

Remark 2.6. As $X_q \leq \Gamma$ which is k -free, $\text{rk } X_q < k$, $C_{q+1} = \langle t \rangle$ is cyclic, and $\text{rk } (X_q \vee C_{q+1}) > \text{rk } X_q = r$, we have $(X_q \vee C_{q+1})$ is the free product of X_q and C_{q+1} by [8, Lemma 4.3].

By the remark and our induction assumption, $\text{rk } \langle X_q, C_{q+1} \rangle = r + 1 \leq (k - 1) + 1 = k$. Therefore $\text{rk } \langle X_q, C_{q+1} \rangle \leq k$, leaving two subcases to consider. First, if $r < k - 1$, then $\text{rk } \langle X_q, C_{q+1} \rangle < k$ and we are done.

In the second subcase, suppose $r = k - 1$. The remark then gives that $\text{rk } \langle X_q, C_{q+1} \rangle = r + 1 = k$; we proceed to prove that $\text{rk } \langle X_q, C_{q+1} \rangle \leq k - 1$ by way of contradiction.

Since $n = q + 1$, by hypothesis $P \in Z_{\log(2k-1)}(C_1) \cap \dots \cap Z_{\log(2k-1)}(C_{q+1})$. Choose a generator γ_i for each C_i , where $1 \leq i \leq q + 1$. For each i , there exists a number $m_i \in \mathbf{N}$ with $d(P, \gamma_i^{m_i} \cdot P) < \log(2k - 1)$ by definition of the cylinders; denote this property (*).

Now the rank of $\langle \gamma_1, \dots, \gamma_{q+1} \rangle$ is k , and so this group is free (being a subgroup of Γ which is k -free). In particular, $\{\gamma_1, \dots, \gamma_{q+1}\}$ is a generating set of a free group of rank k , and so it must contain a subset S of k independent elements whose span has rank k . So let $S = \{\gamma_{i_1}, \dots, \gamma_{i_k}\} \subseteq \{\gamma_1, \dots, \gamma_{q+1}\}$ be as described. Furthermore, the set $S' = \{\gamma_{i_1}^{m_{i_1}}, \dots, \gamma_{i_k}^{m_{i_k}}\}$ is also a set of k independent elements whose span has rank k . Then as $S' \subseteq \text{Isom}_+(\mathbf{H}^3)$ is a set of k freely-generating (loxodromic) generators with $\text{rank}\langle S' \rangle = k$, the $\log(2k - 1)$ Theorem of [3] applies here to give that $\max_{1 \leq j \leq k} d(P, \gamma_{i_j}^{m_{i_j}} \cdot P) \geq \log(2k - 1)$, thereby contradicting property (*) above. Therefore, $\text{rk } \langle X_q, C_{q+1} \rangle \leq k - 1$ as required, and in particular is equal to $k - 1$ in this subcase. \square

We now provide the new notation necessary for setting up the arguments in the remaining sections, as well as a proposition relating the preceding lemma to our new notation. For the next definition, recall Definitions 2.3.

Definition 2.7. Given a point $P \in \mathbf{H}^3$, let $C_P(\lambda)$ denote the set of all C in $C_\lambda(\Gamma)$ for which P is an element of $Z_\lambda(C)$. We then associate to each point P in \mathbf{H}^3 a group, $G_P(\lambda)$, which is defined by $G_P(\lambda) = \langle C : C \in C_P(\lambda) \rangle$. If $C_P(\lambda) = \emptyset$, then set $G_P(\lambda) = \langle 1 \rangle$, and define $\text{rk } G_P(\lambda) = 0$. Also, if the value of λ is understood to be fixed, we may refer to $G_P(\lambda)$ simply as G_P .

Proposition 2.8. *Given $\Gamma \leq \text{Isom}_+(\mathbf{H}^3)$ is discrete, purely loxodromic, and k -free with $k \geq 2$, then for any point $P \in \mathbf{H}^3$, we have $\text{rk } G_P(\log(2k - 1)) \leq k - 1$.*

Proof. This result is a direct consequence of Lemma 2.5 along with the preceding definitions. \square

Definition 2.9. Suppose H is a subgroup of a group G . Then we define the *minimum enveloping rank* of H , or r_H to be the smallest rank among the ranks of groups for which H is a subgroup, if such a number exists. If H is not contained in a finitely generated subgroup of G , then we define r_H to be ∞ . More formally, when H is contained in a finitely generated subgroup K of G , we may define r_H as the smallest positive integer among the set $\{\text{rank } K : H \leq K \leq G\}$.

2.10. Note that if H is non-trivial and non-cyclic, $r_H \geq 2$. Furthermore, if h denotes the rank of H , since H is in particular a subgroup of itself, by definition we have $r_H \leq h$.

Definition 2.11. Suppose $M = \mathbf{H}^3/\Gamma$ is a closed, orientable, hyperbolic 3-manifold ($\Gamma \leq \text{Isom}_+(\mathbf{H}^3)$ is discrete and purely loxodromic). Given a number $\lambda > 0$, we define the number $r_M(\lambda) \in \mathbf{N} \cup \{0\}$ to be the infimum of the set $\{r_{G_P(\lambda)} : P \in \mathbf{H}^3\}$. If the value of λ is understood to be fixed, we may refer to $r_M(\lambda)$ simply as r_M .

Given $M = \mathbf{H}^3/\Gamma$ a closed, orientable, hyperbolic 3-manifold, we now make a few observations regarding the number r_M :

2.12. Given a point P in \mathbf{H}^3 , it follows from the definitions that $r_M \leq r_{G_P(\lambda)} \leq \text{rk } G_P(\lambda)$.

2.13. When $\lambda = \log(2k - 1)$, as a direct consequence of Corollary 2.8 and 2.12, we have $r_M \leq k - 1$.

Remark 2.14. Notice in the standard terminology, saying that the manifold M contains a “ λ -thick” point (i.e. a point of injectivity radius at least $\lambda/2$ in M) is reinterpreted here as saying that $r_M(\lambda) = 0$. We observe that $r_M(\lambda) \neq 0$ if and only if the family of cylinders $\mathcal{Z} = (Z_\lambda(C))_{C \in C_\lambda(\Gamma)}$ forms an open cover of \mathbf{H}^3 .

When $r_M \geq 1$, we claim:

2.15. $\mathbf{H}^3 = \bigcup_{C_1, \dots, C_{r_M} \in C_\lambda(\Gamma)} Z_\lambda(C_1) \cap \dots \cap Z_\lambda(C_{r_M})$.

Proof. Suppose P is a point of \mathbf{H}^3 . As 2.12 says that $\text{rk } G_P \geq r_M$, there exist maximal cyclic subgroups $C_1^P, \dots, C_{r_M}^P$ of Γ such that $\langle C_1^P, \dots, C_{r_M}^P \rangle \leq G_P$ with $P \in Z_\lambda(C_1^P) \cap \dots \cap Z_\lambda(C_{r_M}^P)$ (keeping in mind that P may be in additional cylinders). The statement follows. \square

3. Γ -LABELED COMPLEXES AND CONTRACTIBILITY ARGUMENTS

Definitions 3.1. An indexed covering $\mathcal{U} = (U_i)_{i \in I}$ of a topological space by non-empty open sets defines an abstract simplicial complex called the *nerve of \mathcal{U}* , denoted $K(\mathcal{U})$, whose vertices are in bijective correspondence with the elements of the index set I and whose simplices $\{v_{i_0}, \dots, v_{i_n}\}$ correspond to the non-empty intersections $U_{i_0} \cap \dots \cap U_{i_n}$ of sets of \mathcal{U} . We endow the space which is the geometric realization $|K| = |K(\mathcal{U})|$ with the weak topology. Given a group Γ , a Γ -labeled complex is a pair $(K, (C_v)_v)$ where K is a simplicial

complex and where $(C_v)_v$ is a family of cyclic subgroups of Γ indexed by (and ranging over) the vertices v of K .

Suppose additionally that we are given a positive real number $\lambda > 0$ and subgroup $\Gamma \leq \text{Isom}_+(\mathbf{H}^3)$ which is discrete and cocompact. In particular, if $\mathcal{Z}(\lambda) = (Z_\lambda(C_i))_{i \in I, C_i \in \mathcal{C}_\lambda(\Gamma)}$ is a cover of \mathbf{H}^3 by cylinders, then the family $\mathcal{Z}(\lambda)$ gives rise to a Γ -labeled complex $(K, (C_{v_i})_{v_i})$ where K is the nerve of $\mathcal{Z}(\lambda)$ and where C_{v_i} is the (infinite) maximal cyclic subgroup of Γ that corresponds to the element $Z_\lambda(C_{v_i}) = Z_\lambda(C_i)$ of the cover $\mathcal{Z}(\lambda)$ as indexed by the vertex v_i of K . For purposes of notation, we may refer to this vertex v_i by v_{C_i} .

Definition 3.2. Given a group Γ and $(K, (C_v)_v)$ a Γ -labeled complex, we say the labeling defines a *labeling-compatible* Γ -action on $(K, (C_v)_v)$ if for every vertex v of K , the action defined by $C_{\gamma \cdot v} = \gamma C_v \gamma^{-1}$ is simplicial.

Remark 3.3. Note that if $\Gamma \leq \text{Isom}_+(\mathbf{H}^3)$ is discrete and torsion-free, if the family $\mathcal{Z}(\lambda) = (Z_\lambda(C))_{C \in \mathcal{C}_\lambda(\Gamma)}$ covers \mathbf{H}^3 , and if K is the nerve of $\mathcal{Z}(\lambda)$, then the Γ -labeled complex $(K, (C_v)_v)$ admits a labeling-compatible Γ -action. Let $V = \{v_0, \dots, v_n\}$ be the set of vertices of an n -simplex of K ; by definition $\cap_{0 \leq i \leq n} Z_\lambda(C_{v_i}) \neq \emptyset$. Given $1 \neq \gamma \in \Gamma$ and $v_i \in V$, define $C_{w_i} = C_{\gamma \cdot v_i} = \gamma C_{v_i} \gamma^{-1}$. We must show two things: first, w_i is well-defined as a vertex of K ; or, equivalently, that C_{w_i} is a maximal cyclic subgroup in $\mathcal{C}_\lambda(\Gamma)$, making the action $\gamma \cdot v_i := w_i$ a well-defined action of Γ on the vertices of K ; second, we must show that the set $W = \{w_0, \dots, w_n\}$ of vertices of K is in fact the vertex set of a simplex of K , making this action simplicial, and therefore labeling-compatible. By showing that $\cap_{0 \leq i \leq n} Z_\lambda(C_{w_i})$ is non-empty, we achieve both of these goals.

By our definition, $\cap_{0 \leq i \leq n} Z_\lambda(C_{w_i}) = \cap_{0 \leq i \leq n} Z_\lambda(C_{\gamma \cdot v_i}) = \cap_{0 \leq i \leq n} Z_\lambda(\gamma C_{v_i} \gamma^{-1})$. Using the definition of the cylinders (along with the fact that $\gamma^{-1} \in \text{Isom}_+(\mathbf{H}^3)$ for the first equality), we have $\cap_{0 \leq i \leq n} Z_\lambda(\gamma C_{v_i} \gamma^{-1}) = \cap_{0 \leq i \leq n} \gamma \cdot Z_\lambda(C_{v_i}) = \gamma \cdot \cap_{0 \leq i \leq n} Z_\lambda(C_{v_i})$, which is non-empty as $\cap_{0 \leq i \leq n} Z_\lambda(C_{v_i})$ is non-empty, and hence $\cap_{0 \leq i \leq n} Z_\lambda(C_{w_i}) \neq \emptyset$ as required. Further, since $C_{\gamma^{-1} \cdot w_i} = \gamma^{-1} C_{w_i} \gamma = \gamma^{-1} \gamma C_{v_i} \gamma^{-1} \gamma = C_{v_i}$, the simplex $\{w_0, \dots, w_n\}$ is in fact an n -simplex of K .

Definition 3.4. Given a Γ -labeled complex $(K, (C_v)_v)$ and σ an open simplex in K , define the subgroup $\Theta(\sigma)$ of Γ to be the group $\langle C_v : v \in \sigma \rangle$.

3.5. Suppose K is given to be the nerve of a family $\mathcal{Z}(\lambda) = (Z_\lambda(C_i))_{i \in I, C_i \in \mathcal{C}_\lambda(\Gamma)}$ which is a cover of \mathbf{H}^3 by cylinders. If there exists a point $P \in \mathbf{H}^3$ in the intersection $Z_\lambda(C_0) \cap \dots \cap Z_\lambda(C_n)$, it follows that $\{v_{C_0}, \dots, v_{C_n}\}$ is an n -simplex σ of K , and by the Definitions 2.7 and 3.4, we have $\theta(\sigma) \leq G_P(\lambda)$.

Definitions 3.6. Suppose $(K, (C_v)_v)$ is a Γ -labeled complex. Given an open simplex σ in K , the *minimum enveloping rank* of σ will denote the minimum enveloping rank of the associated subgroup $\Theta(\sigma)$ in Γ . Notice that if $\tau \in K$ is a face of $\sigma \in K$, then we have $r_{\theta(\tau)} \leq r_{\theta(\sigma)}$; i.e. the minimum enveloping rank of a face of σ is less than or equal to that of σ . We may therefore define a subcomplex $K_{(n)}$ of K to be the subcomplex that consists of the non-trivial open simplices σ for which $r_{\theta(\sigma)} \leq n$.

Proposition 3.7. *Suppose K is a simplicial complex, and σ a simplex of K . Suppose further that the link $\text{lk}_K(\sigma)$ is contractible and $X \subset |K|$ is a saturated subset that contains all the simplices for which σ is a face. Then $X - \sigma \hookrightarrow X$ is a homotopy equivalence.*

Proof. Let $C = \cup_{\tau < \sigma} \tau$ be the union of simplices $\tau \in K$ for which σ is a face. By how it is defined, C is homotopy equivalent to $\text{lk}_K(\sigma)$, and is therefore contractible by our assumption. Let $S = \text{star}_K \sigma$. Now $X = (X - \sigma) \cup S$ and $C = (X - \sigma) \cap S$. As C and S are both contractible, then by exactness of the Mayer-Veitoris sequence, the Van Kampen Theorem, and Whitehead's Theorem, it follows that $X - \sigma \hookrightarrow X$ is a homotopy equivalence. \square

Lemma 3.8. *Let $M = \mathbf{H}^3/\Gamma$. Suppose $\mathcal{Z}(\lambda) = (Z_\lambda(C_i))_{i \in I, C_i \in \mathcal{C}_\lambda(\Gamma)}$ is a cover of \mathbf{H}^3 by cylinders and that $r_M \geq k - 2$. Let $|K|$ denote the geometric realization of the nerve of $\mathcal{Z}(\lambda)$. Then $|K| - |K_{(k-3)}|$ is homotopy-equivalent to \mathbf{H}^3 and therefore contractible.*

Proof. The family $(Z_\lambda(C_i))_{i \in I, C_i \in \mathcal{C}_\lambda(\Gamma)}$ covers \mathbf{H}^3 and has the property that every finite intersection of (open) cylinders is contractible, as any such intersection is either empty or convex. Thus Borsuk's Nerve Theorem [4] applies, and we have $|K|$ is homotopy-equivalent to \mathbf{H}^3 . It is only left to show that $|K| - |K_{(k-3)}|$ is homotopy-equivalent to $|K|$. Suppose σ is a non-trivial open simplex of $|K_{(k-3)}|$, which by definition is to say that the minimum enveloping rank of $\theta(\sigma)$ is $\leq k - 3$. Let $v_{i_0}^\sigma, \dots, v_{i_l}^\sigma$ be the vertices of σ , and set $I_\sigma = \{i \in I : v_i \in \sigma\}$.

Let U_i for $i \in I$ denote the cylinder $Z_\lambda(C_i)$ associated with the vertex v_i as defined by the nerve of the cover $\mathcal{Z}(\lambda)$. In particular U_{i_m} will denote the cylinder $Z_\lambda(C_{i_m})$ associated with the vertex $v_{i_m}^\sigma$ of K for $0 \leq m \leq l$. Define the intersection \mathcal{U}_σ to then be $U_{i_0} \cap \dots \cap U_{i_l}$. Let $J_\sigma = \{j \in I - I_\sigma : U_j \cap \mathcal{U}_\sigma \neq \emptyset\}$. Define the set $V_{j,\sigma} = \{U_j \cap \mathcal{U}_\sigma : j \in J_\sigma\}$ and the family $\mathcal{V}_\sigma = (V_{j,\sigma})_{j \in J_\sigma}$.

We proceed to show that:

3.8.1. \mathcal{V}_σ is a cover for \mathcal{U}_σ .

Proof. Suppose on the contrary that \mathcal{V}_σ is in fact *not* a cover for \mathcal{U}_σ . Then there exists a point P of \mathcal{U}_σ such that $P \notin U_i$ for any $i \in I - I_\sigma$. In particular, $G_P(\lambda) \leq \theta(\sigma)$. However by 3.5 we also have $\theta(\sigma) \leq G_P(\lambda)$, and so $\theta(\sigma) = G_P(\lambda)$. Then because $r_{\theta(\sigma)} \leq k - 3$, we have $r_{G_P(\lambda)} \leq k - 3$. But, the minimum enveloping rank of $G_P(\lambda)$ is $\geq k - 2$ as $r_M \geq k - 2$ by hypothesis, providing a contradiction. Therefore, \mathcal{V}_σ covers \mathcal{U}_σ as claimed. \square

So \mathcal{V}_σ , which inherits the subspace topology, is in fact a cover of \mathcal{U}_σ , and so it follows from the definitions that the nerve of \mathcal{V}_σ is simplicially isomorphic to the link of σ in K . Note that two different indices in J_σ may define the same set in \mathcal{V}_σ but they will define different sets in $\mathcal{Z}(\lambda)$; this is why it is essential to define the nerve of \mathcal{V}_σ using J_σ : so that the map from the vertex set of the nerve of \mathcal{V}_σ to the vertex set of the link of σ in K is not only simplicial but bijective; that the inverse of this map is simplicial is straightforward. To see this, suppose v_j is a vertex in the nerve of \mathcal{V}_σ , then by definition $U_j \cap \mathcal{U}_\sigma \neq \emptyset$, i.e. $(U_{i_0} \cap \dots \cap U_{i_l}) \cap U_j$ is non-empty. In particular, $U_{i_0} \cap U_j, U_{i_1} \cap U_j, \dots, U_{i_l} \cap U_j$ are all

non-empty, so that $\{v_{i_0}, v_j\}, \{v_{i_1}, v_j\}, \dots, \{v_{i_l}, v_j\}$ are all edges of K (v_j is distinct from the vertices of σ), and v_j is in the link of σ in K . The reverse inclusion is similar.

Applying Borsuk's Nerve Theorem to \mathcal{V}_σ in place of \mathcal{Z} , we see the underlying space of the nerve of \mathcal{V}_σ is homotopy-equivalent to \mathcal{U}_σ . Since \mathcal{U}_σ is a finite, non-empty intersection of convex open sets, it is contractible. We conclude that the link in K of every simplex of minimum enveloping rank m with $0 \leq m \leq k-3$ is contractible and non-empty.

We now show that the inclusion $|K| - |K_{(k-3)}| \rightarrow |K|$ is a homotopy equivalence.

By local finiteness of the cover \mathcal{Z} from which its nerve $|K|$ is defined, we may index the vertices of $K_{(k-3)}$, and therefore we may index the simplices of $K_{(k-3)}$ and partially order them in the following way: if σ_i, σ_j are such that σ_i is a proper face of σ_j , then $j < i$.

Define $F_n = \sigma_1 \cup \dots \cup \sigma_n$. We may regard $|K| - |K_{(k-3)}|$ as the topological direct limit of the subspaces $K_{F_n} = (|K| - |K_{(k-3)}|) \cup |F_n|$. Thus it suffices to show that the inclusion $K_{F_n} \hookrightarrow K_{F_{n+1}}$ is a homotopy equivalence.

Now $K_{F_{n+1}} - \sigma_{n+1} = K_{F_n}$ and as $\text{lk}_K(\sigma_{n+1})$ is contractible by our work above, we may apply Proposition 3.7 to get $K_{F_{n+1}} - \sigma_{n+1} \cong K_{F_n}$. Hence the inclusion $K_{F_n} \rightarrow K_{F_{n+1}}$ is a homotopy equivalence as required. \square

4. GROUP-THEORETIC PRELIMINARIES

We will say that W is a *saturated* subset of the geometric realization $|K|$ of a simplicial complex K , if W (endowed with the subspace topology) is a union of open simplices of $|K|$ (endowed with the weak topology).

Given a Γ -labeled complex $(K, (C_v)_v)$ and saturated subset $W \subseteq |K|$, we define the subgroup $\Theta(W)$ of Γ to be the group $\langle C_v : v \in \sigma, \sigma \subset W \rangle$.

We now restate Group-Theoretic Conjecture 1.2 from the Introduction which is necessary to prove Proposition 4.3, an essential ingredient in the proof of the Implication Theorem 1.3. Let $H \vee K = \langle H, K \rangle$.

Conjecture 4.1. *Suppose H, K are two rank h subgroups of a free group with $h \geq 3$. If the rank of $H \cap K$ is greater than or equal to h , then the rank of $H \vee K$ must be less than or equal to h .*

Definition 4.2. We say a group Γ has *local rank* $\leq k$ where k is a positive integer, if every finitely generated subgroup of Γ is contained in a subgroup of Γ which has rank less than or equal to k . The local rank of Γ is the smallest k with this property. If there does not exist such a k then we define the local rank of Γ to be ∞ . Note that if Γ is finitely generated, its local rank is simply its rank.

Proposition 4.3. *Assume Conjecture 4.1. Let $k, r \in \mathbf{Z}^+$ with $k > r \geq 3$ and $k \geq 5$. Suppose Γ is a k -free group, $(K, (C_v)_v)$ a Γ -labeled complex, and W a saturated, connected subset of $|K|$ such that $\text{rk } \Theta(\sigma) = r$ for all $\sigma \subset W$. Assume additionally that either*

- (i) there exists a positive integer n such that for all open simplices σ in W , the dimension of σ is n or $n - 1$, or
- (ii) $r = k - 2$ and $\sigma \in |K^{(k-1)}| - |K_{(k-3)}|$ for all $\sigma \in W$.

Then the local rank of $\Theta(W)$ is at most r .

Proof. By definition, we are required to show that every finitely generated subgroup of $\Theta(W)$ is contained in a finitely generated subgroup of $\Theta(W)$ which has rank less than or equal to r . So suppose that $E \leq \Theta(W)$ is a finitely generated subgroup of $\Theta(W)$. Then $E \leq \Theta(V_0)$ for some saturated subset V_0 of W that contains finitely many open simplices. Because W is connected and V_0 contains only finitely many open simplices, there is a smallest connected subset V of W that is a union of finitely many open simplices such that $V_0 \subseteq V$; clearly $E \leq \Theta(V)$ and V is finitely generated. We will show by induction on the number of simplices in V that $\Theta(V)$ has rank at most r .

Proceeding as in [8, Proposition 4.4], by connectedness we may list the (finitely many) open simplices of V in the following way: $\sigma_0, \dots, \sigma_m$, ($m \geq 0$ since V is non-empty) where for any i with $0 \leq i \leq m$, there is an index l with $0 \leq l < i$ such that σ_l is a proper face of σ_i or σ_i is a proper face of σ_l . Define the saturated subset $V_i = \sigma_0 \cup \dots \cup \sigma_i$ for $0 \leq i \leq m$; by induction on i , we will show $\text{rk } \Theta(V_i) \leq r$. The base case is straightforward as $\Theta(V_0) = \Theta(\sigma_0)$ and σ_0 is an open simplex of W , and so has rank r by hypothesis. For the induction step assume $\text{rk } \Theta(V_{i-1}) = r$; we want to show that $\text{rk } \Theta(V_i) = r$. By how we have arranged the list of simplices in V , there is an index l with $0 \leq l < i$ such that σ_l is a proper face of σ_i or σ_i is a proper face of σ_l .

Case (i): First consider the case when σ_i is a proper face of σ_l . Then $\Theta(V_i) = \Theta(V_{i-1})$ as $\sigma_i < \sigma_l \in V_{i-1}$. By our induction assumption, $\text{rk } \Theta(V_{i-1}) \leq r$, and so $\text{rk } \Theta(V_i) \leq r$ as required.

Case (ii): Next, consider the case when σ_l is a proper face of σ_i . Let $P = \Theta(V_{i-1})$, $Q = \Theta(\sigma_i)$ and $R = \Theta(\sigma_l)$. Then $\text{rk } P \leq k - 2$ by the induction hypothesis and $\text{rk } Q = \text{rk } R = k - 2$ by assumption. We want to show $\Theta(V_i) = P \vee Q$ has rank less than or equal to r .

Subcase (i): Assume first that property (i) holds. Then since σ_l is a proper face of σ_i , we must have $\dim \sigma_i = n$ and $\dim \sigma_l = \dim \sigma_i - 1 = n - 1$. Let v denote the vertex of σ_i such that $\text{span}\{\sigma_l, v\} = \sigma_i$ and let $C = C_v$. Then $Q = R \vee C$, and $P \vee C = P \vee Q$. So we proceed to show that $\text{rk}(P \vee C) \leq r$.

By way of contradiction, assume $\text{rk}(P \vee C) > r$. Then since C is infinite cyclic, $P \vee C$ has rank at most $\text{rk } P + 1 = r + 1$ and so $P \vee C$ has rank exactly $r + 1$. As Γ is k -free and $r < k$ (and hence $r + 1 \leq k$), it follows that $P \vee C$ is free as a subgroup of Γ and in particular is the free product of the subgroups P and C ([8, Lemma 4.3]). But, since $R \leq P$, in particular $Q = R \vee C$ is the free product of R and C , and so has rank equal to $\text{rk } R + 1 = r + 1$, which is a contradiction as the rank of Q is exactly r . We conclude that $P \vee C$ has rank $\leq r$ as required for this subcase.

Subcase (ii): Next we assume property (ii). Then $r = \text{rk } Q = \text{rk } R = k - 2$ ($\text{rk } P \leq k - 2$ by induction assumption). As $r = \text{rk } Q = \text{rk } R = k - 2$, both $\dim \sigma_l$ and $\dim \sigma_i$ are at least $k - 3$. Also $\sigma_l, \sigma_i \in K^{(k-1)}$, so both $\dim \sigma_l$ and $\dim \sigma_i$ are $\leq k - 1$. Finally, since our Case (ii)-assumption is that σ_l is a proper face of σ_i , possible pairs $(\dim \sigma_l, \dim \sigma_i)$ are $(k - 3, k - 2)$, $(k - 2, k - 1)$, and $(k - 3, k - 1)$. Let $C \leq \Gamma$ denote the subgroup of Q such that $Q = R \vee C$; then $P \vee Q = P \vee C$ since $R \leq P$.

4.4. First, we look at the rank of P . A priori we know that $\text{rk } P \geq 2$ (i.e. P cannot be cyclic and is non-trivial) since P contains the rank- $(k - 2)$ subgroup R .

In particular, as $R = \Theta(\sigma_l)$ is a subgroup of P , and as σ_l is an element of $K^{(k-1)} - K_{(k-3)}$, we know that the minimum enveloping rank of R is strictly greater than $k - 3$. Along with our induction assumption that $\text{rk } P \leq k - 2$, we conclude the rank of P is exactly $k - 2$. (Note that for this reason in the case when $k = 4$, it is enough only to say in (ii) that $r = 2$, since what is required for the rest of the argument is that P have rank exactly $2 = k - 2$ in this case, an immediate consequence of P containing the rank 2 subgroup R . Specifically, in the $k = 4$ case, we see that a group containing a rank two subgroup certainly cannot have rank one; whereas in cases for $k \geq 5$, one observes that a group that contains a rank three (or more) subgroup *can* have rank two or more, and so that $r_{\theta(\sigma)} \geq k - 2$ is required in the statement of (ii)). Next observe that we must have $\text{rk } C = 1$ or 2 as demonstrated by the possible pairs $(\dim \sigma_l, \dim \sigma_i)$ above. All together, this gives that $\text{rk}(P \vee C) \leq k$ and so $P \vee C$ is free as a subgroup of Γ .

4.5. Next, notice that because $Q \leq P \vee Q$ and Q has minimum enveloping rank $\geq k - 2$, $P \vee Q$ cannot have rank less than $k - 2$. Along with the bound $\text{rk}(P \vee C) \leq k$ of 4.4, we conclude there are only three possibilities for the rank of the group $P \vee Q (= P \vee C)$: these are $k, k - 1$, and $k - 2$.

4.6. As we have $R \leq P$, $R \leq Q$, and $R \leq P \cap Q$, then for the same reason as outlined in 4.4 with $P \cap Q$ taking the place of P , we conclude $\text{rk}(P \cap Q) > k - 3$. Therefore, we may apply Conjecture 4.1 which gives that $\text{rk}(P \vee Q) \leq k - 2$, and so must be equal to $k - 2$ by 4.5, completing this final subcase and proving the proposition.

□

5. THEOREM AND GENERAL BOUND ON r_M

We now restate formally and prove the implicative statement of 1.3 given in the Introduction. For the proof we require a few basic definitions about graphs.

Definitions 5.1. We say that \mathcal{G} is a *graph* if \mathcal{G} is at most a one-dimensional simplicial complex (and so \mathcal{G} has no loops or multiple edges). A *tree* T is a connected graph with no cycles; i.e. T is a graph which is simply connected. Further, if X_i and X_j are disjoint, saturated subsets of a simplicial complex $|K|$, we will make use of the concept of an abstract bipartite graph $\mathcal{G} = \mathcal{G}(X_i, X_j)$ constructed in the following way. Let $\mathcal{W}_i, \mathcal{W}_j$ be the sets of connected components of X_i and X_j respectively. Then the vertices of \mathcal{G} are the elements

of $\mathcal{W}_i \cup \mathcal{W}_j$, and a pair $\{v_{W_i}, v_{W_j}\}$ is an edge if there exist simplices $\sigma \in W_i$ and $\tau \in W_j$ for which $\sigma \leq \tau$ or $\tau \leq \sigma$. Finally, we say that the simplicial action of a group Γ on a graph \mathcal{G} is without inversions if for every $\gamma \in \Gamma$ that stabilizes an edge $e = \{v_1, v_2\} \in \mathcal{G}$, we have $\gamma \cdot v_1 = v_1$ and $\gamma \cdot v_2 = v_2$.

The following two lemmas taken directly from [8] will provide the contradiction necessary to prove Theorem 1.3:

Lemma 5.2. *Suppose that K is a simplicial complex and that X_i and X_j are saturated subsets of $|K|$. Then $|\mathcal{G}(X_i, X_j)|$ is a homotopy-retract of the saturated subset $X_i \cup X_j$ of $|K|$.*

Proof. This is [8, Lemma 5.12]. □

Lemma 5.3. *Let M be a closed, orientable, aspherical 3-manifold. Then $\pi_1(M)$ does not admit a simplicial action without inversions on a tree T with the property that the stabilizer in $\pi_1(M)$ of every vertex of T is a locally free subgroup of $\pi_1(M)$.*

Proof. This is [8, Lemma 5.13] □

Finally, we will appeal to the property stated in the next remark in the proof of Theorem 5.5.

Remark 5.4. Suppose a group Γ admits a labeling-compatible action on a Γ -labeled complex $(K, (C_v))_v$, as is defined in 3.2. If W is a saturated subset of $|K|$ and γ is any element of Γ , it follows that $\Theta(\gamma \cdot W) = \gamma \Theta(W) \gamma^{-1}$. (Since by the definitions, $\Theta(\gamma \cdot W) = \langle C_v : v \in \gamma \cdot W \rangle = \langle C_{\gamma \cdot v} : v \in W \rangle = \langle \gamma C_v \gamma^{-1} : v \in W \rangle = \gamma \langle C_v : v \in W \rangle \gamma^{-1} = \gamma \Theta(W) \gamma^{-1}$). So if an element γ of Γ is invariant on W , then it is in the normalizer of $\Theta(W)$. More generally, the stabilizer in Γ of W is a subgroup of the normalizer of $\Theta(W)$.

The following theorem is the reformulated Implication Theorem 1.3 of the Introduction.

Theorem 5.5. *Suppose M is a closed, orientable, hyperbolic 3-manifold such that $\pi_1(M)$ is k -free with $k \geq 5$. Then if one assumes the Conjecture of 1.2 with $m = k - 2$, setting $\lambda = \log(2k - 1)$ we have $r_M \leq k - 3$.*

Proof. We have $M = \mathbf{H}^3/\Gamma$, where $\Gamma \leq \text{Isom}_+(\mathbf{H}^3)$ is discrete, compact, and torsion-free.

We will assume that $r_M \geq k - 2$ and proceed by way of contradiction. Equivalently, suppose that for all points P in \mathbf{H}^3 , the minimum enveloping rank of G_P is $\geq k - 2$. Then in particular, $\mathbf{H}^3 = \bigcup_{C_1^P, \dots, C_{k-2}^P \in C_{\log(2k-1)}(\Gamma), P \in \mathbf{H}^3} Z_{\log(2k-1)}(C_1^P) \cap \dots \cap Z_{\log(2k-1)}(C_{k-2}^P)$ as described in 2.15. Without loss of generality we write

$\mathbf{H}^3 = \bigcup_{C_1, \dots, C_{k-2} \in C_{\log(2k-1)}(\Gamma)} Z_{\log(2k-1)}(C_1) \cap \dots \cap Z_{\log(2k-1)}(C_{k-2})$, and define the family

$$\mathcal{Z} = (Z_{\log(2k-1)}(C_i))_{C_i \in C_{\log(2k-1)}(\Gamma), 1 \leq i \leq k-2}.$$

We have that \mathcal{Z} is an open cover of \mathbf{H}^3 which satisfies the hypothesis of Lemma 3.8. Then if K denotes the nerve of \mathcal{Z} , the result gives that $|K| - |K_{(k-3)}| \cong \mathbf{H}^3$. Since the inclusion

$|K^{(n)}| - |K_{(k-3)}| \rightarrow |K| - |K_{(k-3)}|$ induces isomorphisms on π_0 and π_1 for $n \geq k-1$ (see [8, Lemma 5.6]), it follows that $|K^{(k-1)}| - |K_{(k-3)}|$ is connected and simply connected.

Let σ be an open simplex in $|K^{(k-1)}| - |K_{(k-3)}|$. Applying Lemma 2.5 with $n = \dim(\sigma) + 1$ (i.e. n is the number of vertices of σ and therefore the number of associated maximal cyclic subgroups of Γ whose associated cylinders have nonempty intersection, as is determined by the nerve), we have that the rank of $\Theta(\sigma)$ is less than or equal to $k-1$. Now since σ is in $|K^{(k-1)}| - |K_{(k-3)}|$, by definition the minimum enveloping rank of $\Theta(\sigma)$ is at least $k-2$. In particular, the rank of $\Theta(\sigma)$ is at least $k-2$ by 2.10.

5.6. All together, we conclude that for any open simplex σ in $|K^{(k-1)}| - |K_{(k-3)}|$, the rank of $\Theta(\sigma)$ is $k-2$ or $k-1$. So, we may write $|K^{(k-1)}| - |K_{(k-3)}|$ as a disjoint union of the saturated subsets X_{k-2} and X_{k-1} , where X_i for $i = k-2, k-1$ is the union of all open simplices σ of $K^{(k-1)}$ for which $\Theta(\sigma)$ has rank i .

We claim the following:

5.6.1. *For $i \in \{k-2, k-1\}$ and for any component W of X_i , the local rank of $\Theta(W)$ is at most i .*

Proof. First we consider the case when $i = k-2$. Then W is a component of X_{k-2} , and for any open simplex σ of X_{k-2} , $\text{rk } \Theta(\sigma)$ is exactly $k-2$. Taking $r = k-2$ in Proposition 4.3, specifically in item (ii), we have that the local rank of $\Theta(W)$ is at most $k-2$. The proof in the case of (ii) shows that the local rank of $\Theta(W)$ is *exactly* $k-2$.

Suppose next that $i = k-1$. Then W is a component of X_{k-1} , and so for each open simplex σ of X_{k-1} , we have $\text{rk } \Theta(\sigma)$ is exactly $k-1$. If d denotes the dimension of σ , then the subgroup $\Theta(\sigma)$ is generated by $d+1$ cyclic groups which are elements of $C_{\log(2k-1)}(\Gamma)$. Hence $\text{rk } \Theta(\sigma) \leq d+1$ and in particular $d \geq \text{rk } \Theta(\sigma) - 1$. As $\text{rk } \Theta(\sigma) = k-1$, we have $d \geq k-2$. But because σ is a simplex contained in $K^{(k-1)}$, d is less than or equal to $k-1$, and so we must have $d = k-2$ or $k-1$. Letting $r = k-1$ and $n = k-1$ in item (i) of Proposition 4.3, we satisfy the hypotheses and the conclusion gives that $\Theta(W)$ has local rank at most $r = k-1$ as desired. \square

Next, we claim:

5.6.2. *The local rank of $\Theta(W)$ is exactly $k-2$ or $k-1$.*

Proof. Let l_W be the local rank of $\Theta(W)$. Our previous claim shows that $l_W \leq k-1$. If in fact $l_W \leq k-3$, then by definition any finitely generated subgroup of $\Theta(W)$ is contained in a finitely generated subgroup of rank less than or equal to $k-3$. As $\Theta(\sigma) \leq \Theta(W)$, this says that $\Theta(\sigma)$ is contained in a subgroup of rank less than or equal to $k-3$ and so the minimum enveloping rank of $\Theta(\sigma)$ would be $\leq k-3$ in this situation. However, given an open simplex σ in W , in particular σ is a simplex of $|K^{(k-1)}| - |K_{(k-3)}|$ and so $\Theta(\sigma)$ has minimum enveloping rank $\geq k-2$, providing a contradiction. Therefore, l_W is $k-2$ or $k-1$. \square

5.6.3. (The analogue of [8, Claim 5.13.2]) *If W is a component of X_{k-2} or X_{k-1} , the normalizer of $\Theta(W)$ in Γ has local rank at most $k - 1$.*

Proof. As a subgroup of Γ , the normalizer of $\Theta(W)$ is k -free. Clearly $\Theta(W)$ is a normal subgroup of its normalizer, and since by the result of 5.6.2 we have $l_W = k - 2$ or $k - 1$ which are strictly less than k , it follows by [8, Proposition 4.5] that the normalizer of $\Theta(W)$ has local rank at most l_W . \square

Set $T = \mathcal{G}(X_{k-2}, X_{k-1})$ (see Definitions 5.1). By Lemma 5.2, T is a homotopy-retract of $X_{k-2} \dot{\cup} X_{k-1}$, which is equal to $|K^{(k-1)}| - |K_{(k-3)}|$ by 5.6. Since $|K^{(k-1)}| - |K_{(k-3)}|$ is connected and simply connected, T is a tree.

By Definition 3.2 of the Γ -labeling compatible action of Γ on K , we see that for any $\gamma \in \Gamma$ and σ in $K^{(k-1)}$, $\Theta(\sigma)$ and $\Theta(\gamma \cdot \sigma)$ are conjugates in Γ (see Remark 5.4), and so have equal rank. Consequently, X_{k-2} and X_{k-1} are invariant under the action of Γ . Note that if w is a vertex of T , the stabilizer Γ_w of w in Γ is really the stabilizer of the associated component W in X_{k-2} or X_{k-1} , and so by Remark 5.4, $\Gamma_w \leq \text{normalizer } \Theta(W)$.

5.7. By our work above in 5.6.3, the local rank of normalizer $\Theta(W)$ is at most $k - 1$, and given that it contains Γ_w as a subgroup, Γ_w must also have local rank at most $k - 1$, and, in particular, is locally free being a subgroup of Γ which is k -free.

Therefore we've constructed an induced action by Γ on the tree T *without* inversions. Since the stabilizer of any vertex of T is locally free as a subgroup of Γ by 5.7, our construction admits a contradiction to Lemma 5.3. \square

The following Propositions and Definitions will be used to explain the geometry of the cases when $r_M(\lambda) = 0$ and 1, and in particular will be used when $\pi_1(M)$ is 5-free and $\lambda = \log 9$ in Corollary 6.8.

Proposition 5.8. *Suppose $\lambda > 0$ and $M = \mathbf{H}^3/\Gamma$ is a closed, orientable hyperbolic 3-manifold with Γ discrete and purely loxodromic. If $r_M = 0$, then M contains an embedded ball of radius $\lambda/2$.*

Proof. As $r_M = 0$, $\text{rk } G_P \geq 0$ for all $P \in \mathbf{H}^3$, and in particular, the choice of r_M means there is a point $P_0 \in \mathbf{H}^3$ with $\text{rk } G_{P_0} = 0$. Then $P_0 \notin Z_\lambda(C)$ for any $C \in C_\lambda(\Gamma)$, and so $d(P_0, \gamma \cdot P_0) \geq \lambda$ for all $\gamma \in \Gamma - \{1\}$, and more generally, $\mathbf{H}^3 \neq \bigcup_{C \in C_\lambda(\Gamma)} Z_\lambda(C)$. If $B_{P_0}(\lambda/2)$ denotes the hyperbolic open ball of radius $\lambda/2$ with center P_0 , in particular this says that the injectivity radius of $B_{P_0}(\lambda/2)$ in M is $\lambda/2$; namely $B_{P_0}(\lambda) \cap \gamma \cdot B_{P_0}(\lambda) = \emptyset$. To see this, consider a point P' in $B_{P_0}(\lambda)$. If in fact it was true that $\gamma(P')$ is also in $B_{P_0}(\lambda)$, it would then follow that $d(P_0, \gamma \cdot P_0) \leq d(P_0, \gamma \cdot P') + d(\gamma \cdot P', \gamma \cdot P_0) < \lambda/2 + \lambda/2 = \lambda$, giving a contradiction. Therefore if $q : \mathbf{H}^3 \rightarrow M$ is the projection map, $q|B : B \rightarrow M$ is injective and the conclusion follows. \square

Definitions 5.9. Let \mathfrak{X}_M be the set of points P in M such that if l_P denotes the length of the shortest, homotopically non-trivial loop based at P , then there is a maximal cyclic

subgroup D_P of $\pi(M, P)$ such that for every homotopically non-trivial loop c based at P of length l_P , we have $[c] \in D_P$. Note that the loop c of length l_P may represent a proper power of a generator of D_P . Let $\mathfrak{s}_M(P)$ be the smallest length of any loop c based at P such that $[c] \notin D_P$.

Proposition 5.10. *Suppose $\lambda > 0$ and $M = \mathbf{H}^3/\Gamma$ is a closed, orientable hyperbolic 3-manifold with Γ discrete and torsion-free. If $r_M = 1$, then there exists a point $P^* \in \mathbf{H}^3$ with $P^* \in \mathfrak{X}_M$ and $\mathfrak{s}_m(P^*) = \lambda$.*

Proof. As $r_M = 1$, the definition of r_M gives that $\text{rk } G_P \geq 1$ for all $P \in \mathbf{H}^3$ (more generally that $\mathbf{H}^3 = \bigcup_{C \in C_\lambda(\Gamma)} Z_\lambda(C)$) and that there is a point $P_0 \in \mathbf{H}^3$ with $\text{rk } G_{P_0} = 1$. Hence $P_0 \in Z_\lambda(C_0)$ for some $C_0 \in C_\lambda(\Gamma)$ and $P_0 \notin Z_\lambda(C)$ for any other $C \in C_\lambda(\Gamma) - C_0$, namely $G_{P_0} = \langle C_0 \rangle$. Set $Z_0 = Z_\lambda(C_0)$ and $Y = \bigcup_{C \in C_\lambda(\Gamma) - C_0} Z_\lambda(C)$. Then $\mathbf{H}^3 = Y \cup Z_0$. Since \mathbf{H}^3 is connected, $(Z_\lambda(C))_{C \in C_\lambda(\Gamma)}$ is an open cover, and Γ is discrete, we must have the intersection $Y \cap Z_0$ is nonempty and open. Notice $P_0 \notin Y$ means $Z_0 \not\subseteq Y$. As Z_0 is connected, we conclude that the frontier of the set $Y \cap Z_0$ relative to Z_0 is nonempty; let F denote this set. Let us choose a point P^* in F . In particular, this says that (i) $P^* \in Z_0$ and (ii) P^* is in the frontier of Y (relative to \mathbf{H}^3). (In concluding (ii), recall that the collection of cylinders in Y comprises a locally finite collection because Γ is *discrete*, and so P^* , a limit point of Y , does not belong to this open collection). If γ_0 is a generator for C_0 , (i) implies that $d(P^*, \gamma_0^m \cdot P^*) < \lambda$ for some integer $m \geq 1$. By (ii), we know that $d(P^*, \gamma_1 \cdot P^*) = \lambda$ for some $\gamma_1 \in \Gamma - C_0$ and that $d(P^*, \gamma \cdot P^*) \geq \lambda$ for all $\gamma \in \Gamma - C_0$. Using the base point $P^* \in \mathbf{H}^3$ to identify $\pi(M, q(P^*))$ with Γ , we have that γ_0^m is represented by a loop of length less than λ based at $q(P^*)$, and any other homotopically non-trivial loop of length less than λ based at $q(P^*)$ is identified with an element of C_0 . Therefore, we have shown the existence of a point $P^* \in \mathfrak{X}_M$ for which the smallest length of any loop represented by $[c]$ in M based at P^* with the property that $[c]$ is not in D_{P^*} , is exactly λ . \square

6. MATRICES AND THEOREM FOR THE CASE $K=5$

We will now restate some of Kent's construction and results regarding joins and intersections of free groups; in particular, we incorporate the background (6.1 and 6.2) as discussed in [10] which is needed to apply [10, Lemma 7] and [10, Lemma 8] to prove Proposition 6.4 that follows. Subsequently, Conjecture 1.2 for rank-3 subgroups H and K is affirmed in Corollary 6.5.

6.1. Given a free group F free on the set $\{a, b\}$, we associate with F the wedge W of two circles based at the wedge point, and we orient the (two) edges of W . Then for any subgroup H of F there is a unique choice of basepoint $*$ in the covering space \widetilde{W}_H such that $\pi_1(\widetilde{W}_H, *)$ is exactly H . Then Γ_H will denote the smallest subgraph containing $*$ of \widetilde{W}_H that carries H . By this construction, Γ_H inherits a natural oriented labeling, i.e. each edge is labeled with one of $\{a, b\}$ and initial and terminal vertices (not necessarily distinct) are determined by the orientation. Hence $\text{rk } \pi_1(\Gamma_H) = \text{rk } H$. Also by this construction, any vertex of Γ_H is at most 4-valent. Define a 3- or more valent vertex of Γ_H to be a *branch vertex*. We will from

here on assume that all graphs in our discussion are normalized so that all branch vertices are 3-valent (see the beginning of [10, Section 3] for this explanation).

6.2. If Γ is a graph, let $b(\Gamma)$ denote the number of branch vertices in Γ . Note that if Γ is 3-regular, i.e. all branch vertices are 3-valent, then we have $\bar{\chi}(\Gamma) = \text{rk}(\pi_1(\Gamma)) - 1 = b(\Gamma)/2$. By 6.1 this says that if H, K are subgroups of F , then $\text{rk}(\pi_1(\Gamma_{H \vee K})) - 1 = \text{rk}(H \vee K) - 1$. If $V(\Gamma_H)$ and $V(\Gamma_K)$ denote the vertex sets of Γ_H and Γ_K respectively, then we can define the graph $\mathcal{G}_{H \cap K}$ whose vertex set is the product $V(\Gamma_H) \times V(\Gamma_K)$ and for which $\{(a, b), (c, d)\}$ is an edge if there are edges $e_1 = \{a, c\}$ in Γ_H and $e_2 = \{b, d\}$ in Γ_K for which e_1 and e_2 have the same label, and e_1 is oriented from a to c and e_2 is oriented from b to d . The graph $\mathcal{G}_{H \cap K}$ is the pullback of the maps $\Gamma_H \rightarrow W$ and $\Gamma_K \rightarrow W$ in the category of oriented graphs, and $\Gamma_{H \cap K}$ is a subgraph of $\mathcal{G}_{H \cap K}$ that carries the fundamental group. We then have the projections $\Pi_H : \mathcal{G}_{H \cap K} \rightarrow \Gamma_H$ and $\Pi_K : \mathcal{G}_{H \cap K} \rightarrow \Gamma_K$. Let the graph \mathcal{T} denote the topological pushout of the maps $\Gamma_{H \cap K} \rightarrow \Gamma_H$ and $\Gamma_{H \cap K} \rightarrow \Gamma_K$ in the category of not properly labeled oriented graphs. Hence the graph \mathcal{T} is defined as the quotient of the disjoint union $\Gamma_H \cup \Gamma_K$ modulo $\sim_{\mathcal{R}}$ where $x \in H$ is equivalent to $y \in K$ if $x \in \Pi_H(\Pi_K^{-1}(y))$ or $y \in \Pi_K(\Pi_H^{-1}(x))$. Now since the map $\mathcal{T} \rightarrow \Gamma_{H \vee K}$ factors into a series of folds (which is surjective at the level of π_1), it follows that $\chi(\mathcal{T}) \leq \chi(\Gamma_{H \vee K})$. Equivalently $\bar{\chi}(\Gamma_{H \vee K}) \leq \bar{\chi}(\mathcal{T})$.

6.3. As outlined in Kent's Section 3.2 [10], we consider the $(2h - 2) \times (2k - 2)$ matrix, $M = (f(x_i, y_j))$, where $f : X \times Y \rightarrow \{0, 1\}$ is the function defined on the sets $X = \{x_1, \dots, x_{2h-2}\}$ and $Y = \{y_1, \dots, y_{2k-2}\}$ of branch vertices of Γ_H and Γ_K , respectively, by letting $f(x_i, y_j) = 1$ if (x_i, y_j) is a branch vertex of $\Gamma_{H \cap K}$ and 0 otherwise. Then the number of ones in M is equal to the number of valence-3 vertices in $\Gamma_{H \cap K}$; i.e. $b(\Gamma_{H \cap K}) = \sum_{i,j} f(x_i, y_j)$. From 6.2, we have $\bar{\chi}(\Gamma_{H \cap K}) = b(\Gamma_{H \cap K})/2$. By [10, Lemma 8], after permuting rows and columns of M , we may write M in the block form: $(M_1, \dots, M_l, O_{(p \times q)})$ where $O_{(p \times q)}$ is the $p \times q$ zero matrix, every row and every column of each of the M_i has a 1. Here, every block M_i of M represents a unique equivalence class of $\sim_{\mathcal{R}}$ with representatives in Γ_H and Γ_K ; all-zero rows represent the $\leq p$ equivalence class(es) of Γ_H which do not have representatives in Γ_K ; and all-zero column(s) represent the $\leq q$ equivalence class(es) of Γ_K which do not have representatives in Γ_H .

Proposition 6.4. $\text{rk}(H \vee K) \leq 1 + \frac{1}{2}(l + p + q)$.

Proof. Note that $2h - 2 \geq l + p$ and $2k - 2 \geq l + q$ as $2h - 2$ is $\#\{\text{rows of } M\}$ and $2k - 2$ is $\#\{\text{columns of } M\}$ where M is the block matrix of 6.3, and so l is bounded above by the positive integer $\min((2h - 2) - p, (2k - 2) - q)$. We have $\text{rk}(H \cup K) - 1 \leq \bar{\chi}(\mathcal{T}) \leq \frac{1}{2}(l + p + q)$ by combining 6.2 and 6.1 along with [10, Lemma 7] for the first inequality and [10, Lemma 8] for the second. In particular, $\text{rk}(H \vee K) \leq 1 + \frac{1}{2}(l + p + q)$ as required. \square

Corollary 6.5. *Suppose $h = k = 3$ and $\text{rk}(H \cap K) \geq 3$. Then $\text{rk}(H \vee K) \leq 3$.*

Proof. As $h = k = 3$, we consider the 4×4 block matrix M of 6.3 where each row and each column of each of the M_i has a 1. So the number of ones, which is the number of valence-3 vertices in $\Gamma_{H \cap K}$, is ≥ 4 .

6.6. Note that l is bounded above by $\min(4 - p, 4 - q)$ by the proof of Proposition 6.4, and so in particular, $l + p \leq 4$ and $l + q \leq 4$. Thus we may rewrite Proposition 6.4 to read $\text{rk}(H \vee K) \leq 1 + \min(2 + \frac{p}{2}, 2 + \frac{q}{2})$. Using this formula, we see that $\text{rk}(H \vee K) \leq 3$ unless p and q are ≥ 2 , and so we need only consider the following cases:

Case $p = 4$ or $q = 4$: This case is an impossibility, as this would imply $M = O_{4 \times 4}$, and hence the number of branch vertices of $\Gamma_{H \cap K}$ is zero, a contradiction, and so we must have $p, q \leq 3$.

Case $p = 3$ or $q = 3$: Suppose first that $p = 3$. Then 6.6 says that $l \leq 1$, implying that $l = 1$ and the top row of M has 4 ones, and so $q = 0$. In this case, the inequality of 6.6 gives $\text{rk}(H \vee K) \leq 1 + \min(2 + \frac{3}{2}, 2)$, and so $\text{rk}(H \vee K) \leq 3$. When $q = 3$ the argument is symmetric, and so the conclusion is satisfied.

Case $p = 2$ or $q = 2$: By symmetry assume $p = 2$. This gives $l \leq 2$ by the bound on l of 6.6. Now if $l \leq 1$, then q must be equal to 3 to satisfy the requirement on the number of ones in M (i.e. the valence-3 vertices in $\Gamma_{H \cap K}$), which is the previous case. Next, if $l = 2$, then as $l \leq \min(4 - p, 4 - q)$, we have $q \leq 2$. First, if $p = q = 2$, then the requirement that the number of ones in M is ≥ 4 fails as the values of p, q , and l would force M to have the form $(M_1, M_2, O_{2 \times 2})$ where $M_1 = M_2 = (1)$, and so M would only contain two ones. Next, if $q = 1$, then again we apply 6.6 to give $\text{rk}(H \vee K) \leq 1 + \min(2 + \frac{2}{2}, 2 + \frac{1}{2}) = 1 + \min(3, 2.5) = 3.5$. Of course, this says that $\text{rk}(H \vee K) \leq 3$ as ranks must be integral and the conclusion is established. \square

We now restate Theorem 1.4 from the Introduction:

Theorem 6.7. *Suppose M is a closed, orientable, hyperbolic 3-manifold such that $\pi_1(M)$ is k -free with $k = 5$. Then when $\lambda = \log 9$, we have $r_M \leq 2$.*

Proof. This is a direct result of Corollary 6.5 along with Theorem 5.5. \square

For the final Corollary recall Definitions 5.9.

Corollary 6.8. *Suppose $M = \mathbf{H}^3/\Gamma$ is a closed, orientable hyperbolic 3-manifold with $\Gamma \leq \text{Isom}_+(\mathbf{H}^3)$ discrete, purely loxodromic and 5-free. Then when $\lambda = \log 9$, one of the following three alternatives holds:*

- (i) M contains an embedded ball of radius $(\log 9)/2$,
- (ii) There exists a point $P^* \in \mathbf{H}^3$ with $P^* \in \mathfrak{X}_M$ such that $\mathfrak{s}_m(P^*)$, is equal to $\log 9$, or
- (iii) $\mathbf{H}^3 = \bigcup_{C_1, C_2 \in \widetilde{C}_{\log 9}(\Gamma)} Z_{\log 9}(C_1) \cap Z_{\log 9}(C_2)$ (i.e. $\text{rk} G_{\tilde{P}} \geq 2$ for all $\tilde{P} \in \mathbf{H}^3$), and there exists a point $\tilde{P} \in \mathbf{H}^3$ such that $\text{rk} G_{\tilde{P}} = 2$.

Let $q : \mathbf{H}^3 \rightarrow \mathbf{H}^3/\Gamma$ be the projection map. As $M = \mathbf{H}^3/\Gamma$, we have $\Gamma \cong \pi_1(M)$. We may then equivalently restate (iii) to say there exists a point $P = q(\tilde{P})$ in M such that the class of all homotopically non-trivial loops of $\pi_1(M, P)$ of length $\leq \log 9$ is contained in a rank-2 subgroup of Γ .

Proof. The result of Theorem 6.7 is that $r_M \leq 2$; so the only possible values for r_M are 0, 1 and 2.

Case (i) follows when $r_M = 0$ and is the result of Proposition 5.8, and Case (ii) follows when $r_M = 1$ and is the result of Proposition 5.10. Case (iii) occurs when $r_M = 2$ and is merely restating that definition. \square

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